RAMAKRISHNA MISSION VIDYAMANDIRA

(A Residential Autonomous College under University of Calcutta)

First Year, Second Semester (January – June), 2011
Mid-Semester Examination, March, 2011

MATHEMATICS (Honours)

Date : 9 March 2011 Full Marks : 50
Time : 11am – 1pm

(Use Separate answer script for each group)

Group – A

1. Answer **any four** questions : [4×3 = 12]

a) Solve the equation : $x^{12} - x^{11} + x^{10} - x^9 + ... + x^2 - x + 1 = 0$

b) Prove that the roots of the equation $z^n = (z+1)^n$ where n is a positive integer are given by $-\frac{1}{2} + i\frac{1}{2}\cot\frac{K\pi}{n}$, K = 1, 2, ..., n – 1.

c) If z is a variable complex number such that $\left|\frac{z-i}{z+1}\right| = K$, show that z lies on a circle in the complex plane if $K \neq 1$ and z lies on a straight line if K = 1.

d) If $a_1, a_2, ..., a_n$ are n positive numbers and if $\left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^n \geq a_1.a_2...a_n$ when n is a power of 2, prove that $\left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^n \geq a_1.a_2...a_n$ when n is not a power of 2. Also deduce that for any positive integer n , $(a_1 \times a_2 \times ... \times a_n)^{\frac{1}{n}} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}$.

e) Prove that 1! 3! 5! ... (2n – 1)! > $(n!)^n$ if n > 1.

f) Find the greatest value of $x^2y^3(6 - x - y)$ when x > 0, y > 0, x + y < 6. Also determine the value of x and y for which the greatest value is attained.

Group – B

2. Answer **any one** question : [3]

a) Prove that every absolutely convergent series of real numbers is convergent.

b) If $\{a_n\}_{n\in\mathbb{N}}$ is a monotone decreasing sequence of positive real numbers converging to zero, show that $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent.

3. Answer **either (a) or (b)**. [10]

a) i) If $\{f(n)\}_{n\in\mathbb{N}}$ be a monotone decreasing sequence of positive real numbers and 'a' be a positive integer (>1), show that the series $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} a^n f(a^n)$ will converge or diverge together.

ii) Let S be the sum of the conditionally convergent series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$, show that the rearranged series $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + ...$, converge to $\frac{3S}{2}$. Give reason for obtaining different sums.

iii) Use Dirichlet's test to show that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ is convergent. [4+4+2]

b) i) Prove that an absolutely convergent series can be expressed as difference of two convergent series of positive real numbers.

ii) Show that the series $\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2+\ldots$ is divergent.

iii) Prove that the following series converges conditionally $\frac{3}{1.2}-\frac{5}{2.3}+\frac{7}{3.4}-\ldots$. [4+3+3]

Group – C

4. Answer **any two** questions : [5+5 = 10]

a) i) Three $n \times n$ matrices A, B, C are such that $AB = I_n$ and $BC = I_n$. Prove that $A = C$

ii) If A be a symmetric matrix of order m and P be an $m \times n$ matrix, prove that P^tAP is a symmetric matrix. [2+3]

b) Prove that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$. [5]

c) i) Prove that an $n \times n$ matrix is invertible if and only if it is non-singular.

ii) If A and B be invertible matrices of the same order then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. [3+2]

Group – D

Answer **any one** question : [15]

5. a) Solve the L.P.P. by graphically :

$$\begin{aligned} &\text{Minimize } z = 20x_1 + 10x_2 \\ &\text{subject to} \quad x_1 + 2x_2 \le 40 \\ &\qquad 3x_1 + x_2 \ge 30 \\ &\qquad 4x_1 + 3x_2 \ge 60 \\ &\qquad x_1, x_2 \ge 0 \end{aligned}$$

b) Prove that the objective function of a linear programming problem assumes its optimal value at an extreme point of the convex set of feasible solutions.

c) If x_1, x_2 be real, prove that the set $X = \{(x_1, x_2) \mid 9x_1^2 + 4x_2^2 \le 36\}$ is a convex set. [5+5+5]

6. a) Solve the L.P.P. by simplex method :

$$\begin{aligned} &\text{Maximize } z = 3x_1 - x_2 \\ &\text{subject to} \quad 2x_1 + x_2 \ge 2 \\ &\qquad x_1 + 3x_2 \le 3 \\ &\qquad x_2 \le 4 \\ &\qquad x_1, x_2 \ge 0 \end{aligned}$$

b) $x_1 = 1, x_2 = 2, x_3 = 1, x_4 = 0$ is a feasible solution of the set of equations :

$11x_1 + 2x_2 - 9x_3 + 4x_4 = 6$

$15x_1 + 3x_2 - 12x_3 + 5x_4 = 9$

Find out the basic feasible solutions and prove that one of them is non-degenerate and other is degenerate.

c) Prove that the set of all convex combinations of a finite number of points is a convex set. [6+5+4]